Local Unitary Invariants for N-qubit Pure State

S. Shelly Sharma*

Depto. de Física, Universidade Estadual de Londrina, Londrina 86051-990, PR Brazil

N. K. Sharma†

Depto. de Matemática, Universidade Estadual de Londrina, Londrina 86051-990 PR, Brazil

The concept of negativity font, a basic unit of multipartite entanglement, is introduced. Transformation properties of determinants of negativity fonts under local unitary (LU) transformations are exploited to obtain relevant N qubit polynomial invariants and construct entanglement monotones, from first principles. It is shown that entanglement monotones that detect the entanglement of specific parts of the composite system may be constructed to distinguish between states with distinct types of entanglement. The structural difference between entanglement monotones for odd and even number of qubits is brought out.

In 1935 Schrödinger [1] coined the term 'entanglement' to describe quantum correlations that make it possible to alter the properties of a distant system instantaneously by acting on a local system. A spin singlet is an example of entangled state of two spin half particles. A qubit is any two level quantum system with basis states represented by $|i\rangle$, $i = 0$ and 1. The spin singlet is an entangled state of two qubits. For a pure state of bipartite quantum system consisting of two distinguishable parts A and B, each of arbitrary dimension, negativity [2] of partially transposed state operator [3] is known to be an entanglement monotone [4]. How properties of one part of a multipartite quantum system are altered by local operations on other parts at distinct remote locations is a complex question. In this letter, we present a novel approach to construct meaningful LU invariants for multi-qubit systems from first principles that is by examining the effect of local unitaries on different parts of the composite system. Our method, illustrated for four qubit case in ref. [5] introduces basic units of entanglement, referred to as negativity fonts. A negativity font is defined as a two by two matrix of probability amplitudes that determines the negative eigen values of a specific four by four submatrix of a partially transposed state operator. It was shown earlier [6] that a partial transpose can be written as a sum of $K-$way ($2 \leq K \leq N$) partial transposes. A $K-$way partial transpose contains information about K body corelations of multipartite system. Contibutions of partial transposes to global negativity, referred to as partial $K-$way negativities are not unitary invariants, but when calculated for canonical states for three qubits [7, 8] and four qubit [9] coincide with entanglement monotones. This article complements our earlier work by outlining a direct method to obtain multiqubit invariants relevant to the construction of entanglement monotones without reaching the canonical state to calculate partial $K-$way negativities. Multi qubit unitary invariants are obtained by examining the transformation properties of negativity fonts present in global partial transpose [3] and $K-$way ($2 \leq K \leq N$) partially transposed matrices [9] constructed from $N-$qubit state operator. The mathematical form of resulting multiqubit invariants for a given state reveals the entanglement microstructure of the state.

Multi qubit invariants, written in terms of determinants of negativity fonts, are essentially relations between intrinsic negative eigenvalues of selected 4×4 submatrices of $K-$way partially transposed matrices. In the case of four qubits, the standard approach from invariant theory has lead to the construction of a complete set of SL-invariants [10, 11] and algorithm for constructing N-qubit invariants is given in ref. [12]. Results for five qubits have also been reported [11]. The N qubit invariants for even number of qubits have been reported earlier in [15] and for even and odd number of qubis in [16]. Focus is on geometric aspects of such invariants in Refs. [17–19]. Independent of these approaches, a method based on expectation values of antilinear operators with emphasis on permutation invariance of the global entanglement measure [13, 14], has been suggested. The number of polynomial invariants is known to increase very fast with number of qubit. However, in general, a small number of invariants is needed to qualify and quantify the entanglement. The advantage of our approach is that it is easily applied to obtain relevant invariants for any state at hand not necessarily the general state or canonical state. Our results bring out the structural difference between LU invariants for $N-$odd and $N-$even qubits through the nature of $K-$way negativity fonts present in respective invariants. For multipartite case, one needs in-equivalent entanglement measures [15, 20, 21]. To show that the method can be used to construct entanglement monotones that detect the entanglement of specific parts of the composite system, four qubit invariants to detect entanglement of a pair of qubits due to four-way correlations are obtained.

---

*Electronic address: shelly@uel.br

†Electronic address: nsharma@uel.br

The entanglement of qubits A_1 and A_2 in pure state $\widehat{\rho}^{A_1 A_2} = \left| \Psi^{A_1 A_2} \right\rangle \left\langle \Psi^{A_1 A_2} \right|$ where

$$\left| \Psi^{A_1 A_2} \right\rangle = a_{00} \left| 00 \right\rangle_{A_1 A_2} + a_{10} \left| 10 \right\rangle_{A_1 A_2} + a_{01} \left| 01 \right\rangle_{A_1 A_2} + a_{11} \left| 11 \right\rangle_{A_1 A_2}, \tag{1}$$

is measured by the negativity of four by four matrix $\left(\rho^{A_1 A_2} \right)_G^{T_1}$ obtained by partially transposing the state of qubit A_1 in $\widehat{\rho}^{A_1 A_2}$. We refer to two by two matrix $\nu^{00} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$ as a negativity font of $\left(\rho^{A_1 A_2} \right)_G^{T_1}$. The squared negativity of $\left(\rho^{A_1 A_2} \right)_G^{T_1}$, is given by $\left(N_G^{A_1} \right)^2 = 4 \left| \det \nu^{00} \right|^2$. If $\det \nu^{00} = 0$, the state is separable. A general N-qubit pure state reads as

$$\left| \Psi^{A_1, A_2, \ldots A_N} \right\rangle = \sum_{i_1 i_2 \ldots i_N} a_{i_1 i_2 \ldots i_N} \left| i_1 i_2 \ldots i_N \right\rangle, \tag{2}$$

where $\left| i_1 i_2 \ldots i_N \right\rangle$ are the basis vectors spanning 2^N dimensional Hilbert space, and A_p is the location of qubit p. The coefficients $a_{i_1 i_2 \ldots i_N}$ are complex numbers. The basis states of a single qubit are labelled by $i_m = 0$ and 1, where $m = 1, \ldots, N$. The global partial transpose of N qubit state $\widehat{\rho} = \left| \Psi^{A_1 A_2 \ldots A_N} \right\rangle \left\langle \Psi^{A_1 A_2 \ldots A_N} \right|$ with respect to qubit p is constructed from the matrix elements of $\widehat{\rho}$ through

$$\left\langle i_1 i_2 \ldots i_N \right| \widehat{\rho}_G^{T_p} \left| j_1 j_2 \ldots j_N \right\rangle = \left\langle i_1 i_2 \ldots i_{p-1} j_p i_{p+1} \ldots i_N \right| \widehat{\rho} \left| j_1 j_2 \ldots j_{p-1} i_p j_{p+1} \ldots j_N \right\rangle. \tag{3}$$

If $\widehat{\rho}$ is a pure state, then the negative eigenvalue of 4×4 sub-matrix of $\widehat{\rho}_G^{T_p}$ in the space spanned by distinct basis vectors $\left| i_1 i_2 \ldots i_p \ldots i_N \right\rangle$, $\left| j_1 j_2 \ldots j_p \ldots j_N \right\rangle$, $\left| i_1 i_2 \ldots j_p \ldots i_N \right\rangle$, and $\left| j_1 j_2 \ldots i_p \ldots j_N \right\rangle$ is $\lambda^- = -\left| \det \left(\nu_K^{i_1 i_2 \ldots i_p \ldots i_N} \right) \right|$ with $\nu_K^{i_1 i_2 \ldots i_p \ldots i_N}$ defined as

$$\nu_K^{i_1 i_2 \ldots i_p \ldots i_N} = \begin{bmatrix} a_{i_1 i_2 \ldots i_p \ldots i_N} & a_{j_1 j_2 \ldots i_p \ldots j_N} \\ a_{i_1 i_2 \ldots j_p \ldots i_N} & a_{j_1 j_2 \ldots j_p \ldots j_N} \end{bmatrix}, \tag{4}$$

where $K = \sum_{m=1}^{N} (1 - \delta_{i_m, j_m})$ $(2 \leq K \leq N)$. In analogy with ν^{00}, 2×2 matrix $\nu_K^{i_1 i_2 \ldots i_p \ldots i_N}$ is defined as a $K-$way negativity font. The subscript K is used to group together the negativity fonts arising due to $K-$way coherences of the composite system that is the correlations responsible for GHZ state like entanglement of a $K-$partite system. For a given value of K, the negativity of $K-$way partial transpose $\widehat{\rho}_K^{T_p}$ with respect to subsystem p, as defined in Ref. [9], arises solely from $K-$way negativity fonts. Determinants of negativity fonts are, in a sense, intrinsic negative eigenvalues of a global or a $K-$way partial transpose of state operator. Global partial transpose of an $N-$qubit state is a combination of $K-$way partially transposed operators $(2 \leq K \leq N)$ [9] and can be expanded as $\widehat{\rho}_G^{T_p} = \sum_{K=2}^{N} \widehat{\rho}_K^{T_p} - (N-2)\widehat{\rho}$. Negativity of $\widehat{\rho}_G^{T_p}$, defined as $N_G^A = \left(\left\| \rho_G^{T_A} \right\|_1 - 1 \right)$, where $\|\widehat{\rho}\|_1$ is the trace norm of $\widehat{\rho}$, arises due to all possible negativity fonts present in $\widehat{\rho}_G^{T_p}$. Since K qubits may be chosen in $\left(\frac{N!}{(N-K)!K!} \right)$ ways the form of a $K-$way font must specify the set of K qubits it refers to. To distinguish between different $K-$way negativity fonts we shall replace subscript K in Eq. (4) by a list of qubit states for which $\delta_{i_m, j_m} = 1$. In other words a $K-$way font involving qubits A_1 to A_K that is $\sum_{m=1}^{N} (1 - \delta_{i_m, j_m}) = \sum_{m=1}^{K} (1 - \delta_{i_m, j_m}) = K$ reads as

$$\nu_{(A_{K+1})_{i_{K+1}}(A_{K+2})_{i_{K+2}} \ldots (A_N)_{i_N}}^{i_1 i_2 \ldots i_p \ldots i_N} = \begin{bmatrix} a_{i_1 i_2 \ldots i_p \ldots i_K i_{K+1} i_{K+2} \ldots i_N} & a_{i_1+1 i_2+1 \ldots i_p \ldots i_K+1 i_{K+1} i_{K+2}, \ldots i_N} \\ a_{i_1 i_2 \ldots i_p+1 \ldots i_K i_{K+1} i_{K+2} \ldots i_N} & a_{i_1+1 i_2+1 \ldots i_p+1 \ldots i_K+1 i_{K+1}, i_{K+2} \ldots i_N} \end{bmatrix}, \tag{5}$$

and its determinant is represented by

$$D_{(A_{K+1})_{i_{K+1}}, (A_{K+2})_{i_{K+2}}, \ldots (A_N)_{i_N}}^{i_1 i_2 \ldots i_p \ldots i_K} = \det \left(\nu_{(A_{K+1})_{i_{K+1}}, (A_{K+2})_{i_{K+2}}, \ldots (A_N)_{i_N}}^{i_1 i_2 \ldots i_p \ldots i_N} \right). \tag{6}$$

Here $i_m + 1 = 0$ for $i_m = 1$ and $i_m + 1 = 1$ for $i_m = 0$. In this notation no subscript is needed for an $N-$way negativity font that is $\nu_N^{i_1 i_2 \ldots i_p \ldots i_N} = \nu^{i_1 i_2 \ldots i_p \ldots i_N}$.

I. TRANSFORMATION OF N-WAY NEGATIVITY FONTS UNDER LOCAL UNITARY ON A SINGLE QUBIT

Determinant of an $N-$way negativity font

$$D^{i_1 i_2 \ldots i_p=0 \ldots i_N} = \det \begin{bmatrix} a_{i_1 i_2 \ldots i_p=0 \ldots i_N} & a_{i_1+1, i_2+1, \ldots i_p=0 \ldots i_N+1} \\ a_{i_1 i_2 \ldots i_p=1 \ldots i_N} & a_{i_1+1, i_2+1, \ldots i_p=1 \ldots i_N+1} \end{bmatrix}, \tag{7}$$

is an invariant of local unitary U^{A_p} acting on qubit A_p. After applying unitary transformation $U^{A_q} = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$ on qubit A_q with $q \neq p$ we obtain

$$U^{A_q} \left| \Psi^{A_1, A_2, \ldots A_N} \right\rangle = \sum_{i_1 i_2 \ldots i_N} b_{i_1 i_2 \ldots i_N} \left| i_1 i_2 \ldots i_N \right\rangle. \tag{8}$$

Using primed symbols for determinants of negativity fonts calculated from coefficients $b_{i_1 i_2 \ldots i_N}$, we can write four transformation equations

$$\left(D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} \right)' = \frac{1}{1+|x|^2} \left[D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} - |x|^2 D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} \right.$$
$$\left. + x D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_0} - x^* D^{i_1 i_2 \ldots i_p=0 \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_1} \right] \tag{9}$$

$$\left(D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} \right)' = \frac{1}{1+|x|^2} \left[D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} - |x|^2 D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} \right.$$
$$\left. + x D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_0} - x^* D^{i_1 i_2 \ldots i_p=0 \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_1} \right] \tag{10}$$

$$\left(D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_0} \right)' = \frac{1}{1+|x|^2} \left[x^* \left(D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} + D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} \right) \right.$$
$$\left. + D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_0} + (x^*)^2 D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_1} \right] \tag{11}$$

$$\left(D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_1} \right)' = \frac{1}{1+|x|^2} \left[x \left(D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} + D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} \right) \right.$$
$$\left. + D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_1} + x^2 D^{i_1 i_2 \ldots i_p=0, \ldots, i_{q-1}, i_{q+1}, \ldots i_N}_{(A_q)_0} \right] \tag{12}$$

relating $N-$way and $(N-1)-$way negativity fonts. Eliminating variable x, invariants of $U^{A_p} U^{A_q}$ are found to be

$$\left(D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} \right)' - \left(D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} \right)' = D^{i_1 i_2 \ldots i_p=0 i_q=0 \ldots i_N} - D^{i_1 i_2 \ldots i_p=0 i_q=1 \ldots i_N}, \tag{13}$$

$$\left[\left(D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} \right)' + \left(D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} \right)' \right]^2 - 4 \left(D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_0} \right)' \left(D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_1} \right)'$$
$$= \left(D^{i_1 i_2 \ldots i_p=0 i_q=0 \ldots i_N} + D^{i_1 i_2 \ldots i_p=0 i_q=1 \ldots i_N} \right)^2 - 4 D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_0} D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_1}, \tag{14}$$

$$\left(D^{i_1 i_2 \ldots i_p=0, i_q=0, \ldots i_N} \right)' \left(D^{i_1 i_2 \ldots i_p=0, i_q=1, \ldots i_N} \right)' - \left(D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_0} \right)' \left(D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_1} \right)'$$
$$= D^{i_1 i_2 \ldots i_p=0 i_q=0 \ldots i_N} D^{i_1 i_2 \ldots i_p=0 i_q=1 \ldots i_N} - D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_0} D^{i_1 i_2 \ldots i_p=0 \ldots i_N}_{(A_q)_1}. \tag{15}$$

Relevant multiqubit invariants for a given value of N can be written down from these general results. Invariants of $U^{A_p} U^{A_r}$ for $K-$way fonts ($2 \leq K \leq N$) with qubits p and r in the superscript and $N-K$ subscripts are analogous to those for $N-$way fonts.

II.   N-EVEN N-WAY INVARIANT

Invariant of $U^{A_1}U^{A_2}U^{A_3}$ is obtained by taking a combination of $N-$way invariants of $U^{A_1}U^{A_2}$ such that Eq. (13) is satisfied for the third qubit, for example

$$I\left(U^{A_1}U^{A_2}U^{A_3}\right) = D^{0000...0} - D^{0100...0} - D^{0010...0} + D^{0110...0}. \tag{16}$$

Using the same reasoning four qubit $N-$way invariant looks like

$$\begin{aligned} I\left(U^{A_1}U^{A_2}U^{A_3}U^{A_4}\right) &= D^{0000...0} - D^{0100...0} - D^{0010...0} + D^{0110...0} \\ &\quad - D^{0001...0} + D^{0101...0} + D^{0011...0} - D^{0111...0}, \end{aligned} \tag{17}$$

and the $N-$way invariant for N qubits reads as

$$I_N = \sum_{i_2..i_N} (-1)^{i_1+i_2...+i_p...+i_N} D^{0i_2...i_N}. \tag{18}$$

Noting that $D^{00i_3...i_N} = -D^{01i_3+1...i_N+1}$, we have

$$\left(D^{00i_3...i_N} + (-1)^{N-1} D^{01i_3+1...i_N+1}\right) = D^{i_1i_2...i_p=0i_q=0...i_N}\left(1+(-1)^N\right), \tag{19}$$

giving $I_{N-odd} = 0$, while for N-even

$$I_{N-even} = \sum_{i_3...i_N} (-1)^{i_3+i_4+...+i_N} D^{00i_3...i_N}. \tag{20}$$

The invariant for $N-$even has permutation symmetry, as such may be used to define $N-$tangle as

$$\tau_{N-even} = 4 \left| \sum_{i_1,i_2,...i_{p-1},i_{p+1}...i_{q-1},i_{q+1},...,i_N} (-1)^{i_1+i_2...+i_p...+i_N} D^{i_1i_2...i_p=0i_q=0...i_N} \right|^2. \tag{21}$$

Degree four invariants for N qubits are obtained by starting with $N-2$ qubit N way invariants and using Eq. (14) to obtain an N qubit invariant.

Four qubit $4-$way invariant with negativity fonts lying solely in $4-$way partial transpose is written from Eq. (20) as $I_4 = D^{0000} + D^{0011} - D^{0010} - D^{0001}$. We identify I_4 with invariant H of degree two as given in ref. [10]. A four qubit state with four qubit entanglement arising due to quantum correlations of the type present in a four qubit GHZ state, is distinguished from other entangled states by a non zero I_4. This entanglement is lost without leaving any residue, on the loss of a single qubit. The entanglement monotone based on I_4 is

$$\tau_4 = 4 \left| \left[D^{0000} + D^{0011} - \left(D^{0010} + D^{0001}\right) \right]^2 \right|,$$

called four-tangle in analogy with three tangle [20]. Four tangle τ_4 vanishes on $W-$like state of four qubits, however, fails to vanish on product of two qubit entangled states.

We now apply the method to construct entanglement monotones that detect the entanglement of specific parts of the composite system, an entangled qubit pair in this case. To obtain degree four invariants that detect products of two qubit states, consider the combination of $4-$way fonts $J = D^{0000} - D^{0100} + D^{0010} - D^{0110}$, which is an invariant of $U^{A_1}U^{A_2}$. Using (Eq. (14)), applied to four-way and three way fonts the four qubit invariant is found to be

$$\begin{aligned} J^{A_1A_2} &= \left(D^{0000} - D^{0100} + D^{0010} - D^{0110}\right)^2 \\ &\quad + 8 D^{00}_{(A_3)_0(A_4)_0} D^{00}_{(A_3)_1(A_4)_1} + 8 D^{00}_{(A_3)_1(A_4)_0} D^{00}_{(A_3)_0(A_4)_1} \\ &\quad - 4 \left(D^{000}_{(A_3)_0} - D^{010}_{(A_3)_0}\right)\left(D^{000}_{(A_3)_1} - D^{010}_{(A_3)_1}\right) \\ &\quad - 4 \left(D^{000}_{(A_4)_0} - D^{010}_{(A_4)_0}\right)\left(D^{000}_{(A_4)_1} - D^{010}_{(A_4)_1}\right). \end{aligned} \tag{22}$$

Similarly, invariant obtained by starting with four-way $U^{A_1}U^{A_3}$ invariant form is

$$
\begin{aligned}
J^{A_1 A_3} = & \left(D^{0000} - D^{0010} + D^{0001} - D^{0011}\right)^2 \\
& + 8D^{00}_{(A_2)_0(A_4)_0}D^{00}_{(A_2)_1(A_4)_1} + 8D^{00}_{(A_2)_1(A_4)_0}D^{00}_{(A_2)_0(A_4)_1} \\
& - 4\left(D^{000}_{(A_2)_0} - D^{010}_{(A_2)_0}\right)\left(D^{000}_{(A_2)_1} - D^{010}_{(A_2)_1}\right) \\
& - 4\left(D^{000}_{(A_4)_0} - D^{001}_{(A_4)_0}\right)\left(D^{000}_{(A_4)_1} - D^{001}_{(A_4)_1}\right),
\end{aligned}
\tag{23}
$$

and starting with $U^{A_1}U^{A_4}$ invariant we get

$$
\begin{aligned}
J^{A_1 A_4} = & \left(D^{0000} - D^{0001} + D^{0010} - D^{0011}\right)^2 \\
& + 8D^{00}_{(A_2)_0(A_3)_0}D^{00}_{(A_2)_1(A_3)_1} + 8D^{00}_{(A_2)_1(A_3)_0}D^{00}_{(A_2)_0(A_3)_1} \\
& - 4\left(D^{000}_{(A_2)_0} - D^{001}_{(A_2)_0}\right)\left(D^{000}_{(A_2)_1} - D^{001}_{(A_2)_1}\right) \\
& - 4\left(D^{000}_{(A_3)_0} - D^{001}_{(A_3)_0}\right)\left(D^{000}_{(A_3)_1} - D^{001}_{(A_3)_1}\right),
\end{aligned}
\tag{24}
$$

with corresponding entanglement monotones defined as $\beta^{A_1 A_i} = \frac{4}{3}\left|J^{A_1 A_i}\right|$, $i = 2 - 4$. By construction $\left|J^{A_1 A_i}\right|$ detects entanglement between qubits $A_1 A_i$, provided the pair $A_1 A_i$ is entangled to its complement in four qubit state. For qubit A_1 the invariants $J^{A_1 A_2}$, $J^{A_1 A_3}$, and $J^{A_1 A_4}$ satisfy the relation $(I_4)^2 = \frac{1}{3}\left(J^{A_1 A_2} + J^{A_1 A_3} + J^{A_1 A_4}\right)$. An interesting four qubit state reported in Ref. [22] is

$$
|\chi\rangle = \frac{1}{\sqrt{8}}\left(|0000\rangle + |1111\rangle - |0011\rangle + |1100\rangle + |1010\rangle - |0101\rangle + |0110\rangle + |1001\rangle\right).
\tag{25}
$$

which is known to have maximal entanglement of the pair $A_1 A_2$ with pair of qubits $A_2 A_4$. The state can be rewritten as an entangled state of $A_1 A_4$ and $A_2 A_3$ Bell pairs

$$
\begin{aligned}
|\chi\rangle = & \frac{1}{\sqrt{8}}\left(|00\rangle_{A_1 A_4} + |11\rangle_{A_1 A_4}\right)\left(|00\rangle_{A_2 A_3} + |11\rangle_{A_2 A_3}\right) \\
& + \frac{1}{\sqrt{8}}\left(|10\rangle_{A_1 A_4} - |01\rangle_{A_1 A_4}\right)\left(|10\rangle_{A_2 A_3} + |01\rangle_{A_2 A_3}\right)
\end{aligned}
$$

however, is not reducible to a pair of Bell states. We verify that for this state $I_4 = 0$, $J^{A_1 A_2} = J^{A_1 A_3} = J^{A_2 A_4} = J^{A_3 A_4} = -\frac{1}{4}$, and $J^{A_1 A_4} = J^{A_2 A_3} = \frac{1}{2}$. Therefore the state is characterised by $\tau_4 = 0$, $\beta^{A_1 A_2} = \beta^{A_1 A_3} = \beta^{A_2 A_4} = \beta^{A_3 A_4} = \frac{1}{3}$, while $\beta^{A_1 A_4} = \beta^{A_2 A_3} = \frac{2}{3}$, indicating that the entanglement of state $|\chi\rangle$ is distinct from that of GHZ state of four qubits having $\tau_4 = 1$, $\beta^{A_1 A_2} = \beta^{A_1 A_3} = \beta^{A_1 A_4} = \frac{1}{3}$ as well as $\beta^{A_2 A_3} = \beta^{A_2 A_4} = \beta^{A_3 A_4} = \frac{1}{3}$.

The degree four invariants for four qubits denoted as L, M, and N in Ref. [11] are combinations of $J^{A_1 A_2}$, $J^{A_1 A_3}$, $J^{A_1 A_4}$ and $(I_4)^2$. Additional invariants are easily constructed to detect all possible types of Four qubit entanglement. One can verify that different types of four qubit entanglement detected by antilinear operators of ref. [13] are quantified by entanglement monotones constructed from four qubit invariants.

III. N-ODD N-WAY INVARIANT

Since $I_{N-odd} = 0$, there is no degree two invariant of $N-$way fonts for a general state of $N-$odd qubits. But we can single out a qubit, write $N - 1$ qubit invariants and then use Eq. (14) to obtain $N-$qubit invariant. If we single out N^{th} qubit and look at negativity fonts of $\rho^{T_{A_1}}$, then two $(N - 1)$ qubit $N-$way invariants are

$$
I^{A_1(A_N)_0}_{N-way} = \sum_{i_3 \dots, i_{N-1}}(-1)^{i_3 + \dots + i_{N-1}}D^{00 i_3 \dots i_{N-1} i_N = 0},
\tag{26}
$$

$$
I^{A_1(A_N)_1}_{N-way} = \sum_{i_3 \dots, i_{N-1}}(-1)^{i_3 + \dots + i_{N-1}}D^{00 i_3 \dots i_{N-1} i_N = 1}.
\tag{27}
$$

Transformation equations for $I^{A_1(A_N)_0}_{N-way}$ and $I^{A_1(A_N)_1}_{N-way}$, under unitary U^{A_N} are written by using Eqs. (9) to (12) and yield an $N-$qubit invariant

$$I_{N-odd}^{A_1 A_N} = \left(I_{N-way}^{A_1 (A_N)_0} + I_{N-way}^{A_1 (A_N)_1}\right)^2 - 4 I_{(N-1)-way}^{A_1 (A_N)_0} \times I_{(N-1)-way}^{A_1 (A_N)_1} \tag{28}$$

with negativity fonts in $\rho^{T_{A_1}}$, where

$$I_{(N-1)-way}^{A_1 (A_N)_0} = \sum_{i_3...,i_{N-1}} (-1)^{i_3+...+i_{N-1}} D_{(A_N)_0}^{00 i_3...i_{N-1}}, \tag{29}$$

and

$$I_{(N-1)-way}^{A_1 (A_N)_1} = \sum_{i_3...,i_{N-1}} (-1)^{i_3+...+i_{N-1}} D_{(A_N)_1}^{00 i_3...i_{N-1}}, \tag{30}$$

are $(N-1)-$way invariants of local unitaries on $(N-1)$ qubits (even no of qubits). Similarly, one may construct $\tau_{N-odd}^{A_1 A_p}$ for $2 \leq p \leq N$ and $N+1 \to 1$ (mod N). Entanglement monotone based on $I_{N-odd}^{A_1 A_p}$ is $\tau_{N-odd}^{A_1 A_p} = 4 \left| I_{N-odd}^{A_1 A_p} \right|$. For $N = 3$, three qubit invariant of degree two determines three-tangle [20], $\tau_3 = 4 \left| \left(D^{000} - D^{001}\right)^2 - 4 D_{(A_2)_0}^{00} D_{(A_2)_1}^{00} \right|$. For $N = 5$, the five-way invariants of local unitaries on qubits A_1, A_2, A_3, and A_4 and corresponding four-way invariants combine to give

$$\begin{aligned}
I_5^{A_1 A_5} &= \left(D^{00000} - D^{00010} + D^{00110} - D^{00100} + D^{00001} - D^{00011} + D^{00111} - D^{00101}\right)^2 \\
&\quad - 4 \left(D_{(A_5)_0}^{0000} - D_{(A_5)_0}^{0001} - D_{(A_5)_0}^{0010} + D_{(A_5)_0}^{0011}\right) \left(D_{(A_5)_1}^{0000} - D_{(A_5)_1}^{0001} - D_{(A_5)_1}^{0010} + D_{(A_5)_1}^{0011}\right)
\end{aligned} \tag{31}$$

which is a five qubit invariant of degree four with fonts in five way and four-way partial transpose with respect to qubit A_1. In general, one can construct $I_5^{A_p A_q}$ obtaining a five tangle $\tau_5^{A_p A_q} = 4 \left| I_5^{A_p A_q} \right|$ for each choice of p and q value. Degree four invariants to detect entanglement of two entangled qubits with their compliment in a five qubit state are combinations of two qubit invariants of five way, four-way, three way and two way fonts and can be obtained in a way analogous to that for five tangle.

To conclude, local unitary polynomial invariants for N qubit quantum state have been obtained from basic units of entanglement, referred to as negativity fonts. The method exploits the transformation properties of determinants of $K-$way negativity fonts under local unitary transformations. The entanglement monotone based on square of degree two invariant for N even (Eq. (21)) and degree four invariant of Eq. (28) for $N-$odd is referred to as $N-$tangle in analogy with three-tangle [20]. The method aims at obtaining LU invariants, that are relevant to classifying multi-qubit entangled states. To illustrate the construction of entanglement monotones that detect entanglement of specific parts of the composite system, degree four invariants to detect entanglement of entangled pairs in a four qubit state are reported. Our method can be used to generate the relevant invariants obtained by using different approaches in references [10–12, 15, 16] and also to generate additional invariants necessary to detect specific entanglement modes. Entanglement monotones constructed from invariants can identify the class to which a given state belongs. Local unitary transformations redistribute the negativity fonts amongst $K-$way partial transposes and may also reduce the number of negativity fonts in a given partial transpose. To determine unitary transformations that relate two unitary equivalent states is an important question in quantum information. The key to determine the unitary transformations relating two states belonging to the same class lies in the numerical value of invariants, number and type of negativity fonts and transformation equations that the determinants of negativity fonts for each state satisfy. The transformation equations for negativity fonts can be used directly to identify the unitaries that relate two unitary equivalent states. The method can be easily extended to qutrits and higher-dimensional systems.

[1] E. Schrödinger, "Discussion of Probability Relations Between Separated Systems," Proceedings of the Cambridge Philosophical Society 31 (1935): 555-563; 32 (1936): 446-451
[2] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A 58, 883 (1998).
[3] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[4] G. Vidal and R. F. Werner, Phys. Rev. Vol. 65, 032314 (2002).
[5] S. S. Sharma and N. K. Sharma, Phys. Rev. A 82, 012340 (2010).

[6] S. S. Sharma and N. K. Sharma, Phys. Rev. A 77, 042117 (2008).
[7] S. S. Sharma and N. K. Sharma, Phys. Rev. A 76, 012326 (2007).
[8] S. S. Sharma and N. K. Sharma, Phys. Rev. A 78, 012113 (2008).
[9] S. S. Sharma and N. K. Sharma, Phys. Rev. A 79, 062323 (2009).
[10] J. G. Luque and J. Y. Thibon, Phys. Rev. A 67, 042303 (2003).
[11] J.-G. Luque and J.-Y. Thibon, J. Phys. A 39, 371 (2006).
[12] J.-G. Luque, J.-Y. Thibon and F. Toumazet, Math. Struct. in Comp. Science, vol. 17, 1133–1151 (2007).
[13] A. Osterloh, and J. Siewert, Phys. Rev. A 72, 012337 (2005).
[14] A. Osterloh, and J. Siewert, Int. J. Quant. Inf. 4, 531 (2006).
[15] A. Wong and N. Christensen, Phys. Rev. A 63, 044301 (2001).
[16] D. Li, X. Li,H. Huang, X. Li, Phys. Rev. A 76, 032304 (2007).
[17] P. Lévay, Phys. Rev. A 71, 012334 (2005).
[18] P. Lévay, J. Phys. A 38, 9075 (2005).
[19] P. Lévay, J. Phys. A 39, 9533 (2006).
[20] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000) .
[21] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[22] Y. Yeo and W. K. Chua, Phys. Rev. Lett. 96, 060502 (2006).